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THE CALCULATION OF GREEN'S FUNCTIONS IN THREE DIMENSIONAL HYDRODYNAMIC GRAVITY WAVE PROBLEMS

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SUMMARY

We have studied the problem of calculating Green's functions in three dimensional hydrodynamic gravity wave problems. A number of new expressions for these functions are presented for both finite and infinite depths. Various techniques for accelerating the convergence of some infinite series in these expressions are investigated and compared with the normal methods of evaluation. A significant improvement in the efficiency of the calculation is found using the results described in this paper.

KEY WORDS Green's Functions Boundary Element Method Waves Special Functions Series Convergence Acceleration

1. INTRODUCTION

The problem of describing the motion of a body in the sea can be formulated in the language of potential theory. If we consider the sea to be an incompressible inviscid fluid with a free surface, the resulting problem involves the solution of Laplace's equation subject to various body, sea floor and free surface boundary conditions. For most geometries it appears impossible to solve these equations analytically, and numerical methods must be used. One technique which has become popular recently is the boundary element method. It has been described extensively by other authors[1–5] so we will not give details but just outline the essential features.

By the use of Green's third identity it can be shown that the potential defining the motion can be reproduced by a distribution of simple sources over the submerged surface of the body. The strengths of the sources can then be found by imposing the body boundary conditions and are given as the solution of a singular integral equation over the submerged body surface. The approximations are then made of (i) taking the submerged surface of the body to consist of a number (n) of simply shaped regions or facets and (ii) assuming that the source strength function has a simple form on each facet. This reduces the integral equation to a system of linear equations, the coefficients of which involve the Green's function for the problem evaluated at a number (m) of points on each facet.

As more and more facets are introduced to improve the accuracy of the calculation, the number of Green's functions which need to be determined grows rapidly ($\sim m^2n^2$) and it is found that this calculation takes an increasingly large proportion of the computing time. Some care is clearly needed in the practical evaluation of these functions in order that the investigation of more complex bodies is not prohibited by the time needed for their calculation.

For a fluid of finite depth, d, the Green's function which satisfies the linearized free surface, fluid bottom and radiation boundary conditions is given by Wehausen and Laitone[6] in the form of

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principal valued integrals or as infinite sums. Suppose (x, y, z), (ξ, η, ζ) are the Cartesian co-ordinates of two points in the fluid where the origin of co-ordinates is taken on the free surface with the positive z direction pointing vertically upwards. If the waves on the free surface have frequency ω then

(i) Infinite depth:

$$G(x, y, z; \xi, \eta, \zeta) = \frac{1}{R} + P\int_0^\infty \frac{(\mu + v)}{(\mu - v)} e^{-\mu\beta} J_0(\mu r) d\mu, \tag{1}$$

where

$$\beta = |z + \zeta|,$$
$$r^2 = (x - \xi)^2 + (y - \eta)^2,$$
$$R^2 = r^2 + (z - \zeta)^2,$$
$$v = \omega^2/g.$$

(ii) Finite depth:

$$G(x, y, z; \xi, \eta, \zeta) = \frac{1}{R} + \frac{1}{R'} + P\int_0^\infty F(\mu) J_0(\mu r) d\mu + i\Lambda J_0(kr), \tag{2}$$

where

$$R'^2 = (z + \zeta + 2d)^2 + r^2,$$
$$F(\mu) = \frac{(\mu + v)e^{-2\mu d} \cosh \mu(z + d) \cosh \mu(\zeta + d)}{(\mu - v) - (\mu + v)e^{-2\mu d}},$$
$$\Lambda = \frac{2\pi(k^2 - v^2) \cosh k(z + d) \cosh k(\zeta + d)}{(k^2 - v^2)d + v},$$
$$v = k \tanh kd.$$

Alternatively

$$G(x, y, z; \xi, \eta, \zeta) = 4 \sum_{m=1}^{\infty} a_m K_m + \Lambda[iJ_0(kr) - Y_0(kr)], \tag{3}$$

where

$$a_m = \frac{(\mu_m^2 + v^2) \cos \mu_m(z + d) \cos \mu_m(\zeta + d)}{(\mu_m^2 + v^2)d - v},$$
$$K_m = K_0(\mu_m r)$$

and

$$-v = \mu_m \tan \mu_m d; \quad \mu_{m+1} > \mu_m > 0, \quad m = 1, 2, \ldots$$

We will refer to (2) as the integral form and (3) as the series form.

The integral forms (1) and (2) are often rather disappointing from a computational point of view.[4,5,7] This is mainly due to the singularity and oscillations in the integrand which makes it necessary to take quite a large number of points in the quadrature formulae used in the estimation of the integral. The evaluation thus becomes relatively rather slow—typically several times slower than the use of the series form—but the use of this form is apparently unavoidable for $r = 0$. Some approximate technique which avoids these problems would clearly be most useful.

The infinite depth case (1) has, consequently, been studied by a number of authors, and our equation (4) in Section 2 has been derived in various ways by Kim,[8] Hearn,[9] Noblesse[10] and Newman.[11] Newman[12] has also given an interesting series expansion of (4) which is valid for any values of r and β. Noblesse[10] has considered the infinite depth case at length and has derived a number of different finite integral, series and asymptotic expansions. These expressions prove to be useful for different parameter ranges (often for 'small' r, β) but they could also provide clues to new ways of writing the Green's function in the finite depth case. Here, relatively little work seems to have been done, although Daubisse[13] has reported some successful numerical calculations in which the integrand of (2) is approximated by a simple pole plus an exponential series.

In many cases the use of the series form (3) represents a very efficient method for calculating the Green's function. Since $\mu_m \to \infty$ as $m \to \infty$, we find that $a_m \to \cos \mu_m(z+d) \cos \mu_m(\zeta + d)/d$ and hence the number of terms of the series which need to be calculated in order to achieve any given accuracy in (3) depends only on the decay of the modified Bessel function $K_0(\mu_m r)$. It can be shown that as $m \to \infty$, $\mu_m \to \mu_m^* = m\pi/d$, so that if we regard $K_0(x)$ as negligible for $x > \bar{x}$, we need to calculate m_0 terms, where $m_0 = \bar{x}d/\pi r$. For many values of (r/d), m_0 is quite acceptable to modern computers, but if (r/d) is small, m_0 can become rather large. It would therefore be useful to find an expression for the infinite sum which converges more quickly for small (r/d).

The results which we wish to present in this paper are organized as follows. In Section 2 we will give an improved integral form of the Green's function for infinite depth. In Section 3 we give a modification of the series form for $r > 0$ which converges more rapidly than (3) and in Section 4 we will derive three new expressions for the Green's function when $r = 0$ in the form of infinite series. Section 5 contains some discussion of the further acceleration of the series forms, and Section 6 gives the results of some calculations using the various forms of the Green's function.

2. THE INTEGRAL FORM—INFINITE DEPTH

Let

$$\begin{aligned} I_\infty(v, \beta, r) &= P\int_0^\infty \frac{(\mu+v)}{(\mu-v)} e^{-\mu\beta} J_0(\mu r) d\mu \\ &= \int_0^\infty e^{-\mu\beta} J_0(\mu r) d\mu + 2vM^-(\beta, r) \\ &= \frac{1}{\sqrt{(r^2+\beta^2)}} + 2vM^-(\beta, r) \end{aligned}$$

where

$$M^\pm(\beta, r) = P\int_0^\infty \frac{e^{-\mu\beta}}{(\mu \pm v)} J_0(\mu r) d\mu.$$

It is easy to show that

$$\frac{\partial M^\pm}{\partial \beta} \mp vM^\pm = -\frac{1}{\sqrt{(r^2+\beta^2)}}$$

and hence that, if $r > 0$

$$M^\pm(\beta, r) = e^{\pm v\beta}\left(M^\pm(0, r) - \int_0^\beta \frac{e^{\mp vx}}{\sqrt{(x^2+r^2)}} dx \right),$$

where

$$M^\pm(0, r) = P\int_0^\infty \frac{J_0(\mu r)}{\mu \pm v} d\mu.$$

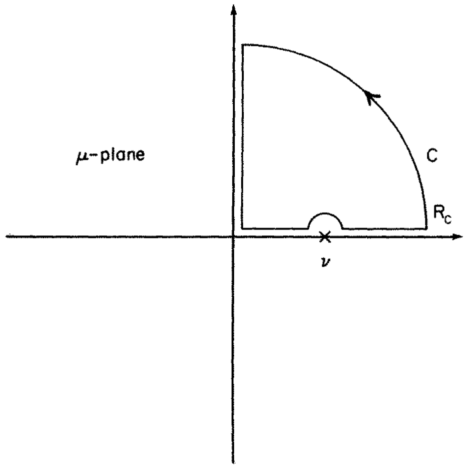

Figure 1. The contour used in the evaluation of $M^{\pm}(0, r)$ in Section 2

In order to evaluate $M^{\pm}(0, r)$ we consider

$$\frac{1}{2\pi i}\int_C \frac{H_0^{(1)}(zr)}{z \pm v}\,dz,$$

where C is the contour shown in Figure 1 and $R_c \to \infty$.

It can be seen that

$$M^{\pm}(0, r) = -\pi Y_0(vr)\varepsilon^{\pm} \pm \frac{2v}{\pi}\int_0^{\infty} \frac{K_0(rt)}{t^2 + v^2}\,dt,$$

where

$$\varepsilon^{+} = 0, \quad \varepsilon^{-} = 1$$

and, hence, from Reference 14 that

$$M^{\pm}(0, r) = \pm\frac{\pi}{2}[H_0(vr) \mp Y_0(vr)],$$

where $H_0(vr)$ is the Struve function of order zero.

Gathering these results we find that for $r > 0$

$$I_{\infty}(v, \beta, r) = \frac{1}{\sqrt{(r^2 + \beta^2)}} - 2ve^{-v\beta}\left\{\int_0^{\beta} \frac{e^{vx}}{\sqrt{(x^2 + r^2)}}\,dx + \frac{\pi}{2}[H_0(vr) + Y_0(vr)]\right\}. \tag{4}$$

When $r = 0$ it is easy to see that $M^{\pm}(\beta, 0) = \pm e^{\pm v\beta}\mathrm{Ei}(\mp v\beta)$ and hence that

$$I_{\infty}(v, \beta, 0) = \frac{1}{\beta} - 2ve^{-v\beta}\mathrm{Ei}(v\beta).$$

It should be noted that the integrals in these formulae involve neither a singular integrand nor a Bessel function and that they may usually be estimated extremely accurately and rapidly by the use of standard quadrature formulae with only a few points.

It is of interest to examine some special cases of (4). First, as $v \to 0$ we find that

$$G \to \frac{1}{R} + \frac{1}{\sqrt{(r^2 + \beta^2)}}. \tag{5}$$

Next, as $\nu \to \infty$

$$G \to \frac{1}{R} + \frac{1}{\sqrt{(r^2+\beta^2)}} - 2\nu e^{-\nu\beta} \int_0^\beta \frac{e^{\nu x}}{\sqrt{(x^2+r^2)}} dx$$

and, using an extension of the Laplace method for evaluating asymptotic expansions of integrals which is given by Erdélyi,[15] we find that as $\nu \to \infty$

$$G \to \frac{1}{R} - \frac{1}{\sqrt{(r^2+\beta^2)}}. \tag{6}$$

Both the forms (5) and (6) agree with the asymptotic results given by Garrison and Berklite.[16]

Finally, we consider the case where $r, \beta \to 0$. That is, we consider the behaviour of the Green's function near a point source on the free surface. Then

$$G \to \frac{1}{R} + \frac{1}{\sqrt{(r^2+\beta^2)}} - 2\nu e^{-\nu\beta} \left(\int_0^\beta \frac{dx}{\sqrt{(x^2+r^2)}} + \log\left(\frac{\nu r}{2}\right) + \gamma \right)$$

where γ is Euler's constant, i.e.

$$G \to \frac{2}{R} - 2\nu e^{-\nu\beta} \left(\log\left\{ \frac{\nu}{2}[\beta + \sqrt{(\beta^2+r^2)}] \right\} + \gamma \right).$$

Thus, for points on the free surface, the Green's function has a double $(1/R)$-type singularity and an additional logarithmic singularity. This result agrees with the observations of Newman.[12]

3. FINITE DEPTH, $r > 0$

One of the techniques which is sometimes employed to accelerate the convergence of an infinite series is to use the known closed form sum of a closely related series. Rather surprisingly this approach can be used in the case of (3). As $m \to \infty$

$$a_m K_m \to \frac{1}{d} \cos \mu_m^*(z+d) \cos \mu_m^*(\zeta+d) K_0(\mu_m^* r) \equiv a_m^* K_m^*$$

and $\sum_{m=1}^{\infty} a_m^* K_m^*$ can be expressed in terms of elementary functions using a result given in Reference 17,

$$\sum_{m=1}^{\infty} \cos mxt\, K_0(mx) = \frac{\pi}{2} \sum_{l=1}^{\infty} \left(\frac{1}{\sqrt{[x^2+(2l\pi - tx)^2]}} + \frac{1}{\sqrt{[x^2+(2l\pi+tx)^2]}} - \frac{1}{l\pi} \right)$$
$$+ \frac{1}{2}\left[\gamma + \log\left(\frac{x}{4\pi}\right) \right] + \frac{\pi}{2x\sqrt{(1+t^2)}}. \tag{7}$$

Applying this result we find, after some manipulation, that

$$G = 4 \sum_{m=1}^{\infty} (a_m K_m - a_m^* K_m^*) + \Lambda[iJ_0(kr) - Y_0(kr)] + \frac{2}{d}\left[\gamma + \log\left(\frac{r}{4d}\right) \right] + \frac{1}{R} + \frac{1}{R'} + \Sigma_G, \tag{8}$$

where

$$\Sigma_G = \sum_{l=1}^{\infty} \left(b_l^+ + b_l^- + c_l^+ + c_l^- - \frac{2}{ld} \right),$$

$$b_l^{\pm} = \{ r^2 + [2ld \pm (z-\zeta)]^2 \}^{-1/2},$$

$$c_l^{\pm} = \{ r^2 + [2ld \pm (z+\zeta+2d)]^2 \}^{-1/2}.$$

It can be shown that for large m

$$a_m K_m - a_m^* K_m^* \sim \frac{v}{m\pi} a_m^* K_m^*,$$

so that this procedure should lead to a significant improvement in the convergence rate. We will discuss this point further in Section 5. The convergence of Σ_G can also be considerably improved by noting that for $l \geqslant 3$

$$b_l^+ + b_l^- + c_l^+ + c_l^- - \frac{2}{ld} \sim \sum_{n=3}^{\infty} \frac{\lambda_n}{l^n},$$

where $\{\lambda_n\}$ are constants. The known values of $\sum_{l=1}^{\infty} 1/l^n$ for $n \geqslant 3$, can then be used to great advantage.

This result can also be derived by modifying the method used by John.[18] He noted that we can write

$$G = P\int_0^{\infty} p(\mu) J_0(\mu r) \mathrm{d}\mu \quad (z > \zeta), \tag{9}$$

where

$$p(\mu) = 2\cosh \mu(\zeta + d) \frac{(\mu \cosh \mu z + v \sinh \mu z)}{(\mu \sinh \mu d - v \cosh \mu d)}.$$

If we write

$$p(\mu) = [p(\mu) - q(\mu)] + q(\mu),$$

where

$$q(\mu) = s(\mu) - \frac{2}{\mu d}$$

and

$$s(\mu) = \frac{2\cosh \mu(\zeta + d) \cosh \mu z}{\sinh \mu d},$$

then we find that

$$G = P\int_0^{\infty} P(\mu) J_0(\mu r) \mathrm{d}\mu + \int_0^{\infty} q(\mu) J_0(\mu r) \mathrm{d}\mu,$$

where

$$P(\mu) = S(\mu) + \frac{2}{\mu d}$$

and

$$S(\mu) = \frac{2v \cosh \mu(z + d) \cosh \mu(\zeta + d)}{(\mu \sinh \mu d - v \cosh \mu d) \sinh \mu d}.$$

$P(\mu)$ has simple poles at $\mu = \pm k, \pm i\mu_m, \pm i\mu_m^*, m = 1, 2, \ldots$ and the first integral can be evaluated using the method described by John together with some results from Section 2, to give

$$P\int_0^{\infty} P(\mu) J_0(\mu r) \mathrm{d}\mu = 4 \sum_{m=1}^{\infty} (a_m K_m - a_m^* K_m^*) - \Lambda J_0(kr)$$

The second integral can be evaluated using the method described in Appendix I to give

$$\int_0^{\infty} q(\mu) J_0(\mu r) \mathrm{d}\mu = \frac{1}{R} + \frac{1}{R'} + \Sigma_G + \frac{2}{d}\left[\gamma + \log\left(\frac{r}{4d}\right)\right]$$

and the expression for G given by equation (8) follows.

John's method can also be modified to produce a new series form for G. As $\mu \to \infty$, $p(\mu) \to e^{\mu(\zeta - z)}(z > \zeta)$ so let us define $\tilde{p}(\mu)$ by

$$p(\mu) = e^{-\mu\beta}\tilde{p}(\mu), \quad \beta = |z - \zeta|.$$

$\tilde{p}(\mu)$ has the same singularity structure as $p(\mu)$ and is bounded for all complex μ provided we exclude an ε-neighbourhood of each pole. It can therefore be expressed in a form analogous to (A12) of Reference 18.

$$\tilde{p}(\mu) = \frac{A^+}{\mu - k} + \frac{A^-}{\mu + k} + 4\sum_{m=1}^{\infty} \frac{\mu A_m - \mu_m B_m}{\mu^2 + \mu_m^2} + \frac{\delta(z, \zeta)}{2}, \tag{10}$$

where

$$A^{\pm} = e^{\pm k\beta}\frac{\Lambda}{\pi}, \quad A_m + iB_m = a_m e^{i\mu_m\beta}.$$

The constant term $\frac{1}{2}\delta(z, \zeta)$ arises from a consideration of the asymptotic behaviour of $\tilde{p}(\mu)$ which is outlined in Appendix III.

Note that letting $\mu = 0$ in (10) we obtain the identity

$$\frac{\Lambda}{2\pi k}\sinh k\beta + \sum_{m=1}^{\infty} \frac{a_m}{\mu_m}\sin \mu_m\beta - \tfrac{1}{8}\delta(z, \zeta) = 0.$$

Following John's method we find that in order to calculate G we need to evaluate the functions $M^{\pm}(\beta, r)$ and $\Delta_m(\beta, r)$
where

$$\Delta_m(\beta, r) = \cos \mu_m\beta \int_0^{\infty} \frac{\mu e^{-\mu\beta}}{\mu^2 + \mu_m^2} J_0(\mu r)d\mu - \sin \mu_m\beta \int_0^{\infty} \frac{e^{-\mu\beta}}{\mu^2 + \mu_m^2} J_0(\mu r)d\mu. \tag{11}$$

$M^{\pm}(\beta, r)$ have already been evaluated in Section 2 so that all that remains is the evaluation of $\Delta_m(\beta, r)$. It is shown in Appendix IV that this can be written in a rather compact way as

$$\Delta_m(\beta, r) = \int_{\beta}^{\infty} \frac{\cos \mu_m x}{\sqrt{(x^2 + r^2)}} dx = K_0(\mu_m r) - \int_0^{\beta} \frac{\cos \mu_m x}{\sqrt{(x^2 + r^2)}} dx.$$

Gathering all these results together we find that

$$G = \frac{\Lambda}{\pi}(e^{k\beta}M^-(\beta, r) + e^{-k\beta}M^+(\beta, r)) + 4\sum_{m=1}^{\infty} a_m\Delta_m(\beta, r) + \frac{1}{2R}\delta(z, \zeta). \tag{12}$$

We note this result here mainly for use in Section 4 because the practical difficulties in evaluating $\Delta_m(\beta, r)$ make this form unsuitable for the calculation of G when $r > 0$.

4. FINITE DEPTH, $r = 0$

Since $K_0(x) \sim -\log(x/2) + \gamma$ for small values of x, the series forms of the Green's function given by (3) and (8) cannot be used when $r = 0$. However, a series form can be derived from (8) by use of the following argument.

Consider the series (8) and integral (2) forms for very small r, and $z \neq \zeta$. Comparing these two we see that

$$I \equiv P\int_0^{\infty} F(\mu)J_0(\mu r)d\mu = 4\sum_{m=1}^{\infty} (a_m K_m - a_m^* K_m^*) + \frac{2}{d}\left[\gamma + \log\left(\frac{r}{4d}\right)\right] + \Sigma_G - \Lambda Y_0(kr).$$

Using the fact that $Y_0(x) \sim (2/\pi)[\log(x/2)+\gamma]$, for small x, we find that for small r

$$I \sim -4\sum_{m=1}^{\infty}(a_m \log \mu_m - a_m^* \log \mu_m^*) + \Sigma_G - \frac{2\Lambda}{\pi}\log k + \frac{2}{d}\log\left(\frac{1}{2d}\right)$$
$$+\left[\gamma + \log\left(\frac{r}{2}\right)\right]\left(-4\sum_{m=1}^{\infty}(a_m - a_m^*) + \frac{2}{d} - \frac{2\Lambda}{\pi}\right).$$

Since it is clear from (2) that I is finite when $r=0$, the coefficient of $\log(r/2)$ in this expression must vanish and we find the identity

$$\sum_{m=1}^{\infty}(a_m - a_m^*) - \frac{1}{2d} + \frac{\Lambda}{2\pi} = 0$$

and that

$$G = \frac{1}{R} + \frac{1}{R'} + \Sigma_G - \frac{2}{d}\log(2d) - \frac{2\Lambda}{\pi}\log k - 4\sum_{m=1}^{\infty}(a_m \log \mu_m - a_m^* \log \mu_m^*). \tag{13}$$

This formula for the Green's function, although obtained here in a somewhat heuristic fashion, can be derived in a more rigorous way by use of contour integration. The technique is to use the expression (9) of the previous section and write

$$p(\mu) = [p(\mu) - q^*(\mu)] + q^*(\mu) \equiv P^*(\mu) + q^*(\mu),$$

where

$$q^*(\mu) = s(\mu) - \frac{2}{\mu(\mu+d)},$$

$$P^*(\mu) = S(\mu) + \frac{2}{\mu(\mu+d)}.$$

It follows that

$$G = P\int_0^{\infty} P^*(\mu)\mathrm{d}\mu + \int_0^{\infty} q^*(\mu)\mathrm{d}\mu \equiv I_1 + I_2.$$

I_1 can now be evaluated by considering the integral $\int_C P^*(-\mu)\log\mu\,\mathrm{d}\mu$ where C is the contour shown in Figure 2. I_2 can be evaluated using the method described in Appendix II.

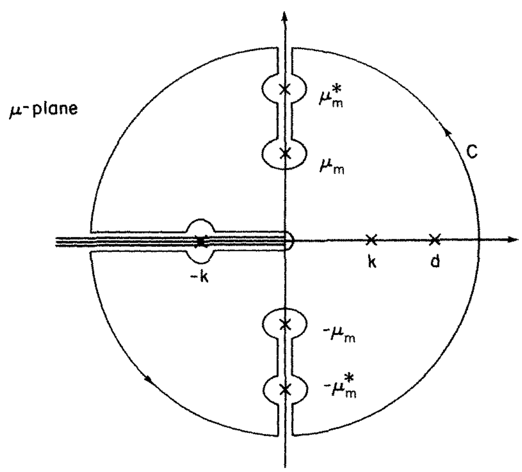

Figure 2. The contour used in the evaluation of I_1 in Section 4

A second expression for G in the case $r=0$ can be obtained from equation (12). When $r=0$, equation (11) gives

$$\begin{aligned}\Delta_m(\beta,0) &= \cos\mu_m\beta\int_0^\infty \frac{\mu e^{-\mu\beta}}{\mu^2+\mu_m^2}\,d\mu-\mu_m\sin\mu_m\beta\int_0^\infty\frac{e^{-\mu\beta}}{\mu^2+\mu_m^2}\,d\mu\\ &= \cos\mu_m\beta g(\mu_m\beta)-\sin\mu_m\beta f(\mu_m\beta)\\ &= -\mathrm{Ci}(\mu_m\beta),\end{aligned}$$

where f, g and Ci are the auxiliary and cosine integral functions defined by Abramowitz and Stegun.[14] In this case we find that

$$G=\frac{\Lambda}{\pi}[E_1(k\beta)-\mathrm{Ei}(k\beta)]-4\sum_{m=1}^{\infty}a_m\mathrm{Ci}(\mu_m\beta)+\frac{1}{2\beta}\delta(z,\zeta). \tag{14}$$

This new result, although of interest, is generally rather inconvenient for the practical evaluation of G owing to the extremely slow decay of the cosine integral function ($\mathrm{Ci}(x)\sim\sin x/x$ for large x). It would be nice if the procedure used in Section 3 could be applied here but we were unable to find any expression for $\sum_{m=1}^{\infty}\cos mxt\,\mathrm{Ci}(mx)$ corresponding to (7). One can be derived however and the derivation is given in Appendix V along with two identities for infinite sums of cosine integral functions analogous to (7). Using the results of Appendix V we find that

$$G=\frac{1}{R}+\frac{1}{R'}-4\sum_{m=1}^{4}(a_m\mathrm{Ci}-a_m^*\mathrm{Ci}^*)+\frac{\Lambda}{\pi}[E_1(k\beta)-\mathrm{Ei}(k\beta)+\Sigma_G+\frac{2}{d}\Big[\gamma+\log\Big(\frac{\beta}{2d}\Big)\Big] \tag{15}$$

where

$$\mathrm{Ci}\equiv\mathrm{Ci}(\mu_m\beta),\quad \mathrm{Ci}^*\equiv\mathrm{Ci}(\mu_m^*\beta).$$

This result can also be obtained by writing, as before in this section, $G=I_1+I_2$ and then evaluating I_1 by a consideration of

$$\int_C\left\{S(-\mu)E_1(\mu\beta)+\frac{2[\log(\mu\beta)+\gamma]}{\mu(\mu-d)}\right\}d\mu,$$

where C is the shown in Figure 2.

Further formulae of this type can be derived in a straightforward way by using the asymptotic behaviour of $\mathrm{Ci}(x)$ to subtract closely related summable series from (14).

5. CONVERGENCE OF SERIES FOR FINITE DEPTH

In previous sections we have given a number of different expressions for the Green's function. Some ((3) and (14)) involve infinite series which are of the form $\sum_{m=1}^{\infty}a_mF_m$ for some function F. We have already commented on the possible difficulties associated with the convergence of these series and we would hope to be able to accelerate this in some way. A second group of formulae ((8), (13) and (15)) involve infinite series of the form $\sum_{m=1}^{\infty}(a_mF_m-a_m^*F_m^*)$ and we would hope that these forms would converge more rapidly, simply as they stand. This is indeed generally true since it is straightforward to show that for 'large' m

$$\mu_m\sim\mu_m^*-\frac{v}{m\pi}$$

and that

$$a_m F_m - a_m^* F_m^* \sim \frac{\nu a_m^*}{m\pi}\left(\frac{\partial F_m}{\partial \mu_m}\right)^* + \frac{\nu F_m^*}{2md}\sum_{\alpha} \alpha \sin \mu_m^* \alpha, \tag{16}$$

where $\alpha = |z - \zeta|$ or $z + \zeta + 2d$.

In the three cases we have to consider the derivative term in this expression decays either at the same rate as F_m or like F_m/m, so that the dominant terms in this difference behave like $(F_m^*/m)\cos\theta_m$ whereas our original series decay as $F_m^*\cos\theta_m$. We do therefore expect some, though not perhaps spectacular, improvement in the convergence of these forms. It would be very helpful if we could find explicit and easily evaluated closed forms for the sums suggested by (16), e.g.

$$\sum_{m=1}^{\infty} \frac{\sin \mu_m^* \theta}{m} K_0(\mu_m^* r)$$

and to use the same subtraction trick again, but we have not been able to find such expressions.

There are, however, many other techniques for accelerating the convergence of infinite series and we have investigated a number of them. One which seems quite effective and simple to implement is described by Keifer and Weiss.[19] Their method emphasizes the trigonometric content of our series, rather than the comparison with a simpler series which we have used so far. They use a simple transformation to write

$$\sum_{m=1}^{\infty} a_m z^m = \frac{1}{[(\rho z)^2 - 2\rho z \cos\psi + 1]}\left[\sum_{m=1}^{\infty} T(a_m) z^{m+1} + a_1\right],$$

where

$$a_m \sim c_m \rho^m \cos(m\psi + \phi_m),$$

$$\lim_{m\to\infty} c_m = 0, \quad \lim_{m\to\infty} \frac{c_{m+1}}{c_m} = 1$$

and

$$T(a_m) = a_{m+1} + \rho^2 a_{m-1} - 2\rho a_m \cos\psi.$$

The transformed series converges more rapidly than the original since $T(a_m) = o(a_m)$ as $m \to \infty$. Higher iterates of the transformation can be generated easily although care must be taken to avoid problems with rounding errors especially near the singularity of the transformation, that is when $\operatorname{Arg} z \simeq \psi$ and $\rho \simeq 1$. We have investigated up to three iterates of this method.

We have also investigated the use of three iterates of the Shanks transformation[20] in its most basic form and the well-known epsilon algorithm of Wynn.[21] Several generalizations of the methods described in References 19 and 20 were also applied but they were not found to be useful in practice.

6. RESULTS

Infinite depth

We have calculated the Green's function for infinite depth using equation (4) with the range of values of ν, β, r corresponding to the parameters X, Y used by Newman.[11] We can confirm the accuracy of the tabulated values of the function given by Newman[11] up to the accuracy of our work, i.e. about 8 decimal places. We have also compared the computing time needed for this calculation with the time required for the straightforward evaluation of (1) using a singularity subtraction technique. The improvement is impressive and speed-up factors in the range 10–60 were obtained. The results did show the importance, from this point of view, of the availability of efficient approximations to the special functions. The use of a recently published approximation for the Struve function $H_0(\nu r)$[22] was particularly beneficial.

Finite depth

In the case of finite depth we have calculated the Green's function using all the methods described above for a range of values of the parameters involved. More precisely we have taken $0{\cdot}1 \leqslant \omega \leqslant 0{\cdot}9, 0 \geqslant z/d, \zeta/d \geqslant -0{\cdot}6, 0 \leqslant r/d \leqslant 0{\cdot}1$. Some typical accurate values are given in Tables I and II.

It should be remembered that in practice we would probably need to calculate the functions several thousand times so that some compromise between accuracy and time must be made. For large m, the mth terms in the infinite sums are of the form $|\varepsilon_m| \cos \theta_m$ where ε_m, θ_m are rather complicated functions of m and the physical parameters involved. The determination of reliable and efficient criteria for terminating the summations is a non-trivial matter since it is rather difficult, efficiently, to control the contribution from the neglected terms, particularly when $z \simeq \zeta$ and (r/d) is small. All the sums discussed in this paper have the same problem, however, so for

Table I. Values of the Green's function for finite depth.

100 ×			(ω)				
$\left\|\frac{z}{d}\right\|$	$\left\|\frac{\zeta}{d}\right\|$	$\frac{r}{d}$	0·1	0·3	0·5	0·7	0·9
10	5	0	0·27907628	0·27441938	0·26570226	0·21008718	0·14465049
10	5	1	0·27504209	0·27036428	0·26159310	0·20589719	0·14041708
10	5	2	0·26417617	0·25943601	0·25050348	0·19456680	0·12895830
10	5	4	0·23295716	0·22797304	0·21840787	0·16153339	0·09545712
10	5	6	0·20225313	0·19688036	0·18630276	0·12794966	0·06124579
10	5	10	0·15708546	0·15059310	0·13707000	0·07459039	0·00697021
20	10	0	0·14443348	0·12848857	0·10365178	0·05974335	0·03672881
20	10	1	0·14391773	0·12796715	0·10311707	0·05920521	0·03621926
20	10	2	0·14241385	0·12644626	0·10155637	0·05763429	0·03473418
20	10	4	0·13697328	0·12093806	0·09588995	0·05192930	0·02937458
20	10	6	0·12950084	0·11335420	0·08804603	0·04402904	0·02205703
20	10	10	0·11333886	0·09684616	0·07073502	0·02660156	0·00654719
40	20	0	0·07722061	0·05282415	0·02853813	0·01714169	0·02171572
40	20	1	0·07715530	0·05275748	0·02847066	0·01708017	0·02165940
40	20	2	0·07696079	0·05255882	0·02826961	0·01689691	0·02149324
40	20	4	0·07620302	0·05178454	0·02748580	0·01618433	0·02084855
40	20	6	0·07500738	0·05056148	0·02624719	0·01506459	0·01984087
40	20	10	0·07165474	0·04712190	0·02276043	0·01195963	0·01708602
60	30	0	0·05612584	0·02913317	0·01228919	0·01409134	0·01907264
60	30	1	0·05610593	0·02911280	0·01226985	0·01407451	0·01905433
60	30	2	0·05604638	0·02905182	0·01221194	0·01402408	0·01900438
60	30	4	0·05581093	0·02881068	0·01198310	0·01382510	0·01880672
60	30	6	0·05542795	0·02841826	0·01161121	0·01350289	0·01848634
60	30	10	0·05427536	0·02723564	0·01049479	0·01254456	0·01753224

Table II. Values of the Green's function for finite depth

$\left\|\frac{z}{d}\right\|$	$100\times\left\|\frac{\zeta}{d}\right\|$	$\frac{r}{d}$	(ω) 0·1	0·3	0·5	0·7	0·9
0	0	1	2·01947218	2·07369636	2·19236611	2·30672742	2·41667376
0	0	2	1·01805534	1·06079968	1·15563185	1·23187986	1·28633334
0	0	4	0·51663302	0·54772106	0·61738813	0·65084183	0·63845681
0	0	6	0·34912775	0·37324739	0·42703189	0·43141443	0·37093130
0	0	10	0·21472024	0·22978733	0·26140355	0·22223695	0·08986071
10	10	1	1·06175328	1·05221243	1·03525000	0·98046298	0·93057487
10	10	2	0·56156251	0·55198522	0·53492859	0·48003429	0·43018074
10	10	4	0·31081984	0·30109861	0·28366963	0·22835571	0·17865628
10	10	6	0·22631643	0·21636142	0·19832701	0·14234632	0·09295728
10	10	10	0·15638160	0·14572425	0·12586708	0·06800241	0·02000885
20	20	1	1·03562655	1·01569065	0·98895108	0·95796052	0·95108355
20	20	2	0·53560098	0·51565536	0·48889746	0·45792229	0·45110563
20	20	4	0·28549937	0·26551498	0·23868416	0·20777126	0·20119567
20	20	6	0·20199900	0·18195045	0·15499943	0·12419281	0·11801851
20	20	10	0·13481649	0·11456620	0·08724155	0·05679516	0·05189983
40	40	1	1·02335033	0·99696357	0·97831970	0·97721863	0·98258920
40	40	2	0·52334550	0·49695678	0·47831502	0·47722425	0·48259753
40	40	4	0·27332620	0·24692965	0·22829634	0·22724677	0·23263069
40	40	6	0·18996078	0·16355123	0·14493211	0·14395111	0·14935240
40	40	10	0·12319216	0·09674129	0·07816837	0·07740598	0·08286009
60	60	1	1·02258902	0·99506671	0·98543648	0·99074050	0·99387588
60	60	2	0·52258433	0·49506168	0·48543483	0·49074093	0·49387539
60	60	4	0·27256562	0·24504155	0·23542825	0·24074270	0·24387331
60	60	6	0·18920119	0·16167477	0·15208407	0·15741236	0·16053652
60	60	10	0·12243571	0·09490185	0·08538349	0·09075564	0·09385903

comparison of these methods we have simply terminated the summation when $|\varepsilon_m| < \varepsilon$ for a range of values of ε. Further work on this point is in progress and the results will be presented elsewhere.

It is obvious that the computing time involved in each method depends crucially both on the efficient coding of the convergence acceleration techniques and of the algorithms for calculating the special functions involved. In this work we have not attempted to optimize these aspects of the calculation but have chosen, instead, to compare the various methods by simply counting the number of 'special' function evaluations required in each, i.e. K_0, Ci, log.

Some typical results are shown in Tables III and IV together with data relating to the evaluation of the Green's function using equation (3). It can be seen that the new methods which we have described in this paper are often very effective in improving the efficiency of these calculations. In some cases there is little to choose between the various methods but the best method does seem to be almost independent of desired accuracy. Generally speaking we have to work harder, i.e. use more terms in higher iterates, as r decreases and ω increases, but in most cases the new methods are superior to the straightforward use of (3).

For modest accuracies equation (8) is a sound basis particularly at low frequencies but in general the accelerated versions of (3) using the method of Keifer and Weiss[19] work best when $r \neq 0$, $z \neq \zeta$.

Table III. n_ε is the number of 'special' function calculations needed to determine the Green's function neglecting contributions of modulus less than ε using method Meth $= MA$. $M = 1$ refers to the use of equation (3), $M = 3$ refers to (15) and $M = 4$ to (13). $A = K$ suggests the use of the convergence acceleration technique of Keifer and Weiss,[19] $A = \mathrm{W}$ implies Wynn's epsilon algorithm.[21] N_ε is the number of terms used with (3).

	100 ×					n_ε ω ($\varepsilon = 0{\cdot}00001$)						n_ε ω ($\varepsilon = 0{\cdot}000001$)			
$\left\|\frac{z}{d}\right\|$	$\left\|\frac{\zeta}{d}\right\|$	$\frac{r}{d}$	Meth	0·1	0·3	0·5	0·7	0·9	N_ε	0·1	0·3	0·5	0·7	0·9	N_ε
10	5	0	3W	42	70	76	88	92		58	82	98	102	110	
10	5	1	1K	36	40	48	48	49	239	54	56	58	58	58	308
10	5	2	1K	31	31	32	31	30	120	37	36	35	38	48	154
10	5	4	1K	25	25	24	23	22	60	28	28	28	30	29	78
10	5	6	1K	20	19	20	20	19	40	26	25	25	24	24	52
10	5	10	1K	14	14	14	14	14	24	18	18	18	18	16	31
20	10	0	3K/4K	32	42	52	54	54		50	54	62	94	100	
20	10	1	1K	26	26	26	28	28	239	41	41	41	40	40	308
20	10	2	1K	22	22	21	26	26	120	30	31	32	32	32	154
20	10	4	1K	18	18	16	16	21	60	20	26	27	27	27	78
20	10	6	1K	15	15	15	15	16	40	19	19	18	23	23	52
20	10	10	1K	12	12	12	11	12	24	15	15	15	15	15	31
40	20	0	3W/4K	22	28	38	42	42		46	46	46	66	66	
40	20	1	1K	16	16	16	21	21	239	24	24	26	27	31	308
40	20	2	1K	15	16	16	16	18	120	20	22	24	26	26	154
40	20	4	1K	13	14	15	15	15	60	17	17	21	21	21	78
40	20	6	1K	11	11	13	14	14	40	14	17	16	16	19	52
40	20	10	1K	10	11	10	12	12	24	12	11	14	15	15	31
60	30	0	3W/4K	18	28	30	32	34		30	34	36	38	46	
60	30	1	1K	12	16	16	15	18	239	18	23	22	24	22	308
60	30	2	1K	12	14	14	15	15	120	18	20	19	19	22	154
60	30	4	1K	10	10	12	13	15	60	16	16	19	19	19	78
60	30	6	1K	10	10	12	12	13	40	16	16	14	15	15	52
60	30	10	1K	9	10	10	11	11	24	13	12	12	13	13	31

Equation (8) is useful however when $z = \zeta$ and (r/d) is small. This is because the singularity of the transformation of Reference 19 mentioned earlier reduces the effectiveness of the normal method and (8), combined with the acceleration techniques which do not try to exploit the oscillatory nature of the terms, i.e. References 20 and 21, becomes more effective.

For the case $r = 0$, the slow decay of the functions involved in (13), (14) and (15) means that we must use more terms than for the case $r > 0$. It also proved impossible to use the acceleration technique of Reference 19 with equations (14) and (15) since the relationship between the arguments of the trigonometric and cosine integral functions means that we are again at a singularity of the transformation. Various modifications of the method used to derive (14) were attempted in order to avoid this problem but they did not prove to be helpful. For small $|z - \zeta|/d$ and small ω, the best approach seems to be the use of equation (15) with the epsilon algorithm[21] but, as these two quantities increase, equation (13) together with Reference 19 is more powerful.

Table IV. n_ε is the number of 'special' function calculations needed to determine the Green's function neglecting contributions of modulus less than ε using method Meth $= MA$. $M = 1$ refers to the use of equation (3), $M = 2$ refers to (8). $A =$ K suggests the use of the convergence acceleration technique of Keifer and Weiss[19], $A =$ W implies Wynn's epsilon algorithm,[21] and $A =$ S implies Shank's transformation.[20] N_ε is the number of terms used with (3).

	100 ×					n_ε ω						n_ε ω			
$\left\|\frac{z}{d}\right\|$	$\left\|\frac{\zeta}{d}\right\|$	$\frac{r}{d}$	Meth	0·1	0·3	0·5	0·7	0·9	N_ε	0·1	0·3	0·5	0·7	0·9	N_ε
0	0	1	2S	18	22	26	32	38	239	28	32	34	44	60	308
0	0	2	2S	16	20	24	24	28	120	20	28	36	40	40	154
0	0	4	2S	16	18	20	20	26	60	20	22	24	24	28	78
0	0	6	1S/W	14	14	14	16	16	40	18	18	18	20	20	52
0	0	10	1S/W	11	11	12	12	14	24	13	13	13	14	14	31
10	10	1	2/1W	38	54	82	88	90	239	70	96	96	104	110	308
10	10	2	2/1W	38	38	56	70	70	120	42	70	82	88	86	154
10	10	4	1K	30	32	30	28	29	60	32	31	36	38	38	78
10	10	6	1K	20	22	22	22	23	40	26	26	29	29	29	52
10	10	10	1K	14	14	15	15	15	24	18	18	18	20	20	31
20	20	1	2/1W	44	50	66	78	84	239	58	82	96	96	88	308
20	20	2	2/1W	34	44	54	52	50	120	38	56	60	60	66	154
20	20	4	1W	30	30	32	34	36	60	38	38	36	36	38	78
20	20	6	1W	22	24	26	28	25	40	28	28	30	27	30	52
20	20	10	1K	16	16	16	16	16	24	17	17	20	20	21	31
40	40	1	2W	32	46	50	62	62	239	50	66	74	90	86	308
40	40	2	2/1W	26	38	36	40	40	120	38	44	48	48	46	154
40	40	4	1W	22	28	28	30	26	60	30	32	32	36	34	78
40	40	6	1W	22	24	24	24	24	40	26	26	26	28	26	52
40	40	10	1W	16	18	18	20	18	24	20	22	20	22	20	31
60	60	1	2W	30	44	58	58	66	239	46	64	74	90	96	308
60	60	2	1W	30	38	38	36	36	120	38	48	46	52	52	154
60	60	4	1W	28	28	30	28	26	60	32	32	32	32	34	78
60	60	6	1W	18	24	22	22	22	40	26	26	28	26	26	52
60	60	10	1W	18	18	18	18	18	24	20	20	20	22	22	31
						$\varepsilon = 0{\cdot}00001$						$\varepsilon = 0{\cdot}000001$			

7. CONCLUSIONS

We have presented a number of alternative expressions for the Green's functions in hydrodynamic wave problems and some identities involving the cosine integral functions. In many cases the calculation of Green's functions can be considerably improved using these forms, particularly when it is associated with one of the convergence acceleration techniques discussed.

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APPENDIX I

In this appendix we will outline a method of evaluation of $\int_0^\infty q(\mu)J_0(\mu r)\mathrm{d}\mu$ which was needed in Section 3. Using the result[17]

$$\int_0^\infty \mathrm{e}^{-\phi\mu}J_0(\mu r)\mathrm{d}\mu = \frac{1}{\sqrt{(r^2+\phi^2)}}, \quad \phi, r > 0, \tag{17}$$

we can write

$$\frac{1}{\sqrt{[r^2+(2ld\pm\alpha)^2]}} - \frac{1}{2ld} = \int_0^\infty \mathrm{e}^{-2ld\mu}[\mathrm{e}^{\mp\alpha\mu}J_0(\mu r) - 1]\mathrm{d}\mu, \quad 2ld \pm \alpha \geqslant 0.$$

Hence

$$\begin{aligned}\Sigma_G &= \sum_{l=1}^{\infty}\int_0^\infty 2\mathrm{e}^{-2ld\mu}\{[\cosh\mu(z-\zeta)+\cosh\mu(z+\zeta+2d)]J_0(\mu r)-2\}\mathrm{d}\mu \\ &= 4\int_0^\infty \sum_{l=1}^{\infty}\mathrm{e}^{-2ld\mu}[\cosh\mu(z+d)\cosh\mu(\zeta+d)J_0(\mu r)-1]\mathrm{d}\mu \\ &= 2\int_0^\infty \frac{\mathrm{e}^{-\mu d}}{\sinh\mu d}[\cosh\mu(z+d)\cosh\mu(\zeta+d)J_0(\mu r)-1]\mathrm{d}\mu\end{aligned}$$

and, after some manipulation, again using (17), we find

$$\begin{aligned}\frac{1}{R}+\frac{1}{R'}+\Sigma_G &= \int_0^\infty (\mathrm{e}^{-\mu(z-\zeta)}+\mathrm{e}^{-\mu(z+\zeta+2d)})J_0(\mu r)+\Sigma_G \\ &= \int_0^\infty \left[\left(q(\mu)+\frac{2}{\mu d}\right)J_0(\mu r) - \frac{2\mathrm{e}^{-\mu d}}{\sinh\mu d}\right]\mathrm{d}\mu.\end{aligned}$$

Thus

$$\begin{aligned}\int_0^\infty q(\mu)J_0(\mu r)\mathrm{d}\mu &= \frac{1}{R}+\frac{1}{R'}+\Sigma_G+2\int_0^\infty\left(\frac{\mathrm{e}^{-\mu d}}{\sinh\mu d}-\frac{J_0(\mu r)}{\mu d}\right)\mathrm{d}\mu \\ &= \frac{1}{R}+\frac{1}{R'}+\Sigma_G+2\int_0^\infty\left(\frac{\mathrm{e}^{-\mu d}}{\sinh\mu d}-\frac{\mathrm{e}^{-2\mu d}}{\mu d}\right)\mathrm{d}\mu \\ &\quad -\frac{2}{d}\int_0^\infty\left(\frac{J_0(\mu r)-\mathrm{e}^{-2\mu d}}{\mu}\right)\mathrm{d}\mu.\end{aligned}$$

Now, using the results[17]

$$\int_0^\infty \frac{J_0(x)-\mathrm{e}^{-\alpha x}}{x}\mathrm{d}x = \log(2\alpha), \quad \alpha > 0,$$

$$\int_0^\infty\left(\frac{1}{1-\mathrm{e}^{-x}}-\frac{1}{x}\right)\mathrm{e}^{-x}\mathrm{d}x = \gamma,$$

the result given in Section 3 follows.

APPENDIX II

In this appendix we will outline a method of evaluation of $\int_0^\infty q^*(\mu)\mathrm{d}\mu$ which was needed in Section 4.

Following the method of Appendix I with $r=0$ we have

$$\int_0^\infty q^*(\mu)\mathrm{d}\mu = \frac{1}{R}+\frac{1}{R'}+\Sigma_G+2\int_0^\infty\left(\frac{\mathrm{e}^{-\mu d}}{\sinh\mu d}-\frac{1}{\mu(\mu+d)}\right)\mathrm{d}\mu$$

$$=\frac{1}{R}+\frac{1}{R'}+\Sigma_G+2\int_0^\infty\left(\frac{\mathrm{e}^{-\mu d}}{\sinh\mu d}-\frac{\mathrm{e}^{-2\mu d}}{\mu d}\right)\mathrm{d}\mu-2\int_0^\infty\left(\frac{1}{\mu(\mu+d)}-\frac{\mathrm{e}^{-2\mu d}}{\mu d}\right)\mathrm{d}\mu$$

and with the use of the result[17]

$$\int_0^\infty\left(\frac{1}{1+\alpha x}-\mathrm{e}^{-2x}\right)\frac{\mathrm{d}x}{x}=\gamma-\log\left(\frac{\alpha}{2}\right),\quad \alpha>0,$$

we find that

$$\int_0^\infty q^*(\mu)\mathrm{d}\mu=\frac{1}{R}+\frac{1}{R'}+\Sigma_G+\frac{2}{d}\log\left(\frac{1}{2d^2}\right).$$

APPENDIX III

In this appendix we will outline a method of calculation of the function $\delta(z,\zeta)$ defined in equation (10) of Section 3.

It is easy to see by considering $p(\mu)$ that $\lim_{\mu\to\infty}\tilde{p}(\mu)=\rho(z,\zeta)$ where

$$\rho(z,\zeta)=\begin{cases}1 & 0>z\geqslant\zeta>-d\\ 2 & z=0 \text{ or } \zeta=-d\\ 4 & z=0 \text{ and } \zeta=-d.\end{cases}$$

Also,

$$\lim_{\mu\to\infty}\tilde{p}(\mu)=\tfrac{1}{2}\delta(z,\zeta)+\lim_{\mu\to\infty}\sum_{m=1}^{\infty}\frac{\mu A_m-\mu_m B_m}{\mu^2+\mu_m^2}$$

$$=\tfrac{1}{2}\delta(z,\zeta)+\lim_{\mu\to\infty}\sum_{m=1}^{\infty}\frac{\mu A_m^*-\mu_m^* B_m^*}{\mu^2+\mu_m^2}.$$

Now,

$$\sum_{m=1}^{\infty}\frac{\mu A_m^*-\mu_m^*B_m^*}{\mu^2+\mu_m^2}=\frac{d}{\pi}\sum_{m=1}^{\infty}\{\mu[\cos 2\mu_m^*(z+d)+\cos 2\mu_m^*(\zeta+d)+\cos 2\mu_m^*(z-\zeta)+1]$$

$$-\mu_m^*[\sin 2\mu_m^*(z+d)-\sin 2\mu_m^*(\zeta+d)+\sin 2\mu_m^*(z-\zeta)]\}\Big/\left(m^2+\frac{\mu^2d^2}{\pi^2}\right).$$

If we consider the results[17]

$$\beta\sum_{m=1}^{\infty}\frac{\cos m\alpha}{m^2+\beta^2}=\frac{\pi}{2}\frac{\cosh\beta(\pi-\alpha)}{\sinh\beta\pi}-\frac{1}{2\beta},\quad 0\leqslant\alpha\leqslant 2\pi,$$

$$\sum_{m=1}^{\infty}\frac{m\sin m\alpha}{m^2+\beta^2}=\frac{\pi}{2}\frac{\sinh\beta(\pi-\alpha)}{\sinh\beta\pi},\quad 0<\alpha<2\pi,$$

$$=0,\quad \alpha=0 \text{ or } 2\pi,$$

then

$$\lim_{\beta\to\infty} \beta \sum_{m=1}^{\infty} \frac{\cos m\alpha}{m^2+\beta^2} = 0, \quad 0<\alpha<2\pi,$$

$$= \frac{\pi}{2}, \quad \alpha = 0 \text{ or } 2\pi,$$

$$\lim_{\beta\to\infty} \sum_{m=1}^{\infty} \frac{m \sin m\alpha}{m^2+\beta^2} = 0.$$

Applying these results we find that

$$\lim_{\mu\to\infty} \sum_{m=1}^{\infty} \frac{\mu A_m^* - \mu_m^* B_m^*}{\mu^2+\mu_m^{*2}} = \begin{cases} \frac{1}{2}, & 0>z>\zeta>-d, \\ 1, & 0=z>\zeta>-d, \\ & 0>z>\zeta=-d, \\ & 0>z=\zeta>-d, \\ 2, & z=\zeta=0, \\ & z=\zeta=-d, \\ & z=0 \text{ and } \zeta=-d, \end{cases}$$

and comparing this result with the formula for $\rho(z,\zeta)$ we obtain the result

$$\delta(z,\zeta) = \begin{cases} 1, & 0>z>\zeta<-d, \\ 2, & 0=z>\zeta>-d, \\ 2, & 0>z>\rho=-d, \\ 4, & 0=z>\zeta=-d, \\ 0, & \text{otherwise}. \end{cases}$$

APPENDIX IV

In this appendix we will outline a method of simplification of the function $\Delta_m(\beta, r)$ defined in equation (11).

If we define

$$I(\beta) \equiv \int_0^{\infty} \frac{\mu e^{-\mu\beta} J_0(\mu r) d\mu}{\mu^2+\mu_m^2}, \quad J(\beta) \equiv \int_0^{\infty} \frac{e^{-\mu\beta} J_0(\mu r) d\mu}{\mu^2+\mu_m^2},$$

it is easy to show using (17) that

$$\frac{dI}{d\beta} = -\frac{1}{\sqrt{r^2+\beta^2}} + \mu_m^2 J$$

$$\frac{dJ}{d\beta} = -I$$

and hence that

$$\frac{d^2J}{d\beta^2} + \mu_m^2 J = \frac{1}{\sqrt{(r^2+\beta^2)}}.$$

This equation can be solved for J in a straightforward manner and used to give an expression for

I. Now

$$\Delta_m(\beta, r) = I \cos \mu_m \beta - \mu_m J \sin \mu_m \beta$$
$$= -A\mu_m - \int_0^\beta \frac{\cos \mu_m x\mathrm{d}x}{\sqrt{(r^2+x^2)}}, \quad A \text{ constant}$$

and since[17] $\Delta_m(0, r) = K_0(\mu_m r) = -A\mu_m$

$$\Delta_m(\beta, r) = K_0(\mu_m r) - \int_0^\beta \frac{\cos \mu_m x}{\sqrt{(r^2+x^2)}}\mathrm{d}x$$
$$= \int_\beta^\infty \frac{\cos \mu_m x\mathrm{d}x}{\sqrt{(r^2+x^2)}}.$$

APPENDIX V

In this appendix we will give a derivation of expressions for some infinite sums of cosine integral functions analogous to equation (7) which were used in Section 4.

Consider the behaviour of the expressions for G given by (13) and (14) as $v \to 0$. In this case $k^2 \sim v/d$, so that $E_1(k\beta) - \mathrm{Ei}(k\beta) \sim -2\log(k\beta) + \gamma$, $\Lambda \sim \pi/d$. Hence, from (13)

$$G \to \frac{1}{z-\zeta} + \frac{1}{z+\zeta+2d} + \Sigma_G - \frac{2}{d}\log(2d) - \frac{2}{d}\log k$$

and from (14)

$$G \to -\frac{2}{d}\log(k\beta) - \frac{2}{d}\gamma - 4\sum_{m=1}^{\infty} a_m^* \mathrm{Ci}^* + \frac{\delta(z,\zeta)}{2\beta}.$$

These expressions are, of course, equal and comparing them we find that

$$-4\sum_{m=1}^{\infty} a_m^* \mathrm{Ci}^* = \frac{1}{z-\zeta} + \frac{1}{z+\zeta+2d} + \frac{2}{d}\left[\log\left(\frac{z-\zeta}{2d}\right) + \gamma\right] + \Sigma_G - \frac{\delta(z,\zeta)}{2(z-\zeta)}. \tag{18}$$

If we let $\zeta = -d$ and write $z + d = \beta d$, (18) gives

$$-4\sum_{m=1}^{\infty} \cos m\pi\beta \mathrm{Ci}(m\pi\beta) = \frac{1}{\beta} + 2\left[\gamma + \log\left(\frac{\beta}{2}\right)\right] + 2\sum_{l=1}^{\infty}\left(\frac{1}{2l+\beta} + \frac{1}{2l-\beta} - \frac{1}{l}\right) \tag{19}$$

and when $\beta = 1$,

$$-\sum_{m=1}^{\infty} (-1)^m \mathrm{Ci}(m\pi) = \gamma + \log\left(\frac{1}{2}\right) + \sum_{l=1}^{\infty}\left(\frac{1}{2l+1} + \frac{1}{2l-1} - \frac{1}{l}\right). \tag{20}$$

If we note that

$$a_m^* = \frac{1}{2d}\left[\cos\frac{m\pi}{d}(z-\zeta) + \frac{\cos m\pi}{\alpha}(z+\zeta+2d)\right]$$

and write $z + \zeta + 2d = \alpha d$, then combining (18) and (19) gives

$$-2\sum_{m=1}^{\infty} \cos m\pi\alpha \mathrm{Ci}(m\pi\beta) = \frac{1}{\alpha} + \gamma + \log\left(\frac{\beta}{2}\right) + \sum_{l=1}^{\infty}\left(\frac{1}{2l+\alpha} + \frac{1}{2l-\alpha} - \frac{1}{l}\right), \quad 0 < \alpha < 2, \quad 0 < \beta < 1. \tag{21}$$

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